Lec 35: 11/18/2009

When Identical Particles Are Practically Distinguishable?

So far we have discussed the fundamental difference between identical particles in classical and quantum physics. I.E, the fact that they are not distinguishable, thus symmetrization or antisymmetrization of the total wave function.

However, we know that in many cases we can treat them (partially) as distinguishalbe particles. For example, doing experiments with a beam of electrons, we do not need to know about all the electrons in the universe. Only those in the beam should be take into account.

So, the question is that how this comes about. To elucidate, lets consider two identical particles, one

in the state $|\psi_1\rangle$ and the other in the state $|\psi_2\rangle$. Th
probability distribution to find one particle at point x_1 and the other at point x_2 is given by:

$$P(x_1, x_2) = \left| \frac{1}{2} \langle x_1 x_2| + \langle x_2 x_1| \; |\psi_1 \psi_2\rangle + |\psi_2 \psi_1\rangle \right|^2 \quad \text{(bosons)}$$

$$P(x_1, x_2) = \left| \frac{1}{2} \langle x_1 x_2| - \langle x_2 x_1| \; |\psi_1 \psi_2\rangle - |\psi_2 \psi_1\rangle \right|^2 \quad \text{(fermions)}$$

Note that we have used symmetric or antisymmetric combinations of $|\psi_1\rangle \otimes |\psi_2\rangle$, as well as $|x_1\rangle \otimes |x_2\rangle$, depending on whether the particles are bosons or fermions.

Thus:

$$P_B(x_1, x_2) = |\psi_1(x_1)\psi_2(x_2) + \psi_1(x_2)\psi_2(x_1)|^2$$

$$P_F(x_1, x_2) = |\psi_1(x_1)\psi_2(x_2) - \psi_1(x_2)\psi_2(x_1)|^2$$

Expanding these, we find:

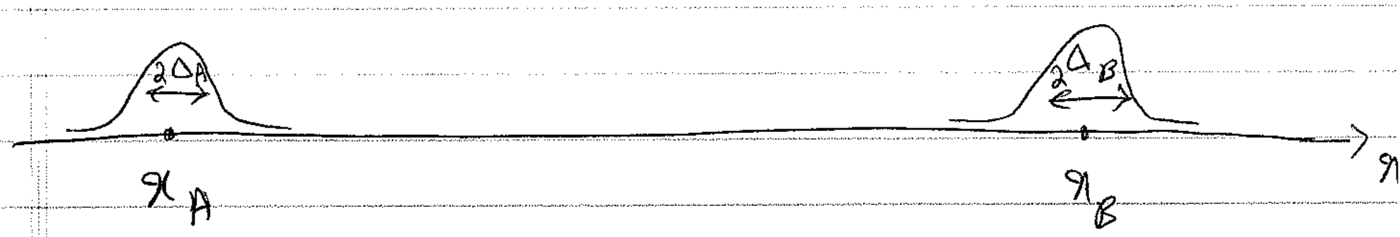
$$P_B(x_1, x_2) = |\psi_1(x_1)|^2 |\psi_2(x_2)|^2 + |\psi_1(x_2)|^2 |\psi_2(x_1)|^2 +$$

$$2\,\mathrm{Re}\left[\psi_1(x_1)\psi_2(x_2)\psi_1^*(x_2)\psi_2^*(x_1)\right] \quad (1)$$

$$P_F(x_1, x_2) = |\psi_1(x_1)|^2 |\psi_2(x_2)|^2 + |\psi_1(x_2)|^2 |\psi_2(x_1)|^2 -$$
$$2\,\mathrm{Re}\left[\psi_1(x_1)\psi_2(x_2)\psi_1^*(x_2)\psi_2^*(x_1)\right] \quad (2)$$

Notice that the difference between bosons and fermions manifests itself in the sign of last term on the right-hand sign of above expressions.

Now lets assume that ψ_1 and ψ_2 are two wavepackets centered at two far away points x_A and x_B:



$\psi_1(x)$ rapidly decreases for x outside the $[x_A-\Delta_A, x_A+\Delta_A]$ interval, while $\psi_2(x)$ rapidly decreases for x outside the $[x_B-\Delta_B, x_B+\Delta_B]$ interval.

Now we find the probability to find one of the particles near x_A ($x_1 \in [x_A - \Delta_A, x_A + \Delta_A]$) and the other near x_B ($x_2 \in [x_B - \Delta_B, x_B + \Delta_B]$).

For two bosons, the only term in ① that is significant will be:

$$P_B^1(x_A, x_2) \simeq |\Psi_1(x_A)|^2 |\Psi_2(x_2)|^2$$

Since the other terms are small (they have $\Psi_1(x_2)$ and $\Psi_2(x_1)$) and can be neglected.

Another possibility is that $x_2 \in [x_A - \Delta_A, x_A + \Delta_A]$ and $x_1 \in [x_B - \Delta_B, x_B + \Delta_B]$, hence:

$$P_B^2(x_A, x_1) \simeq |\Psi_1(x_A)|^2 |\Psi_2(x_1)|^2$$

$$P_{B(\bar{x}_A)} = P_B^1(x_1, x_2) + P_B^2(x_1, x_2) = |\Psi_1(x_A)|^2 |\Psi_2(x_2)|^2 + |\Psi_2(x_1)|^2 |\Psi_1(x_A)|^2$$

The total probability distribution to find one particle near x_A, regardless of the position of the second particle is:

$$\frac{1}{2}\int_{-\infty}^{+\infty} dx_2 \, |\psi_1(x_A)|^2 \, |\psi_2(x_2)|^2 + \frac{1}{2}\int dx_1 \, |\psi_1(x_A)|^2 |\psi_2(x_1)|^2$$

The factor of $\frac{1}{2}$ is important since $P(x_1, x_2)$ satisfies:

$$\frac{1}{2}\int P(x_1, x_2)\, dx_1\, dx_2 = 1$$

This is because of the identical nature of particles. When we integrate over x_1 and x_2, we must divide by 2 in order to avoid overcounting. Note that having a particle at $x_1 = a$ and the other at $x_2 = b$ is physically the same as $x_1 = b$ and $x_2 = a$. All that matters is that one particle is at a and one is at b.

Now we use the fact that ψ_1 and ψ_2 are normalized wavefunctions. Thus:

$$P_B(x_A) = \frac{1}{2}|\psi_1(x_A)|^2 + \frac{1}{2}|\psi_1(x_A)|^2 = |\psi_1(x_A)|^2$$

Interestingly, we find the same thing in the case of fermions;

$$P_F(r_A) \approx |\Psi_1(r_A)|^2$$

Therefore, if the distance between particles is much larger than the spread in their wavefunction, they can be effectively treated as distinguishable particles.

In general, de Broglie wavelength is a measure of the spread in the wavefunction. Thus, as long as the average distance between particles is larger than the de Broglie wavelength, a system of identical bosons effectively behave as a system of identical fermions, and both are effectively similar to a system of distinguishabe particles.

This is provided that the average distance betwee particles remains larger than the de Broglie wavelength at all times.

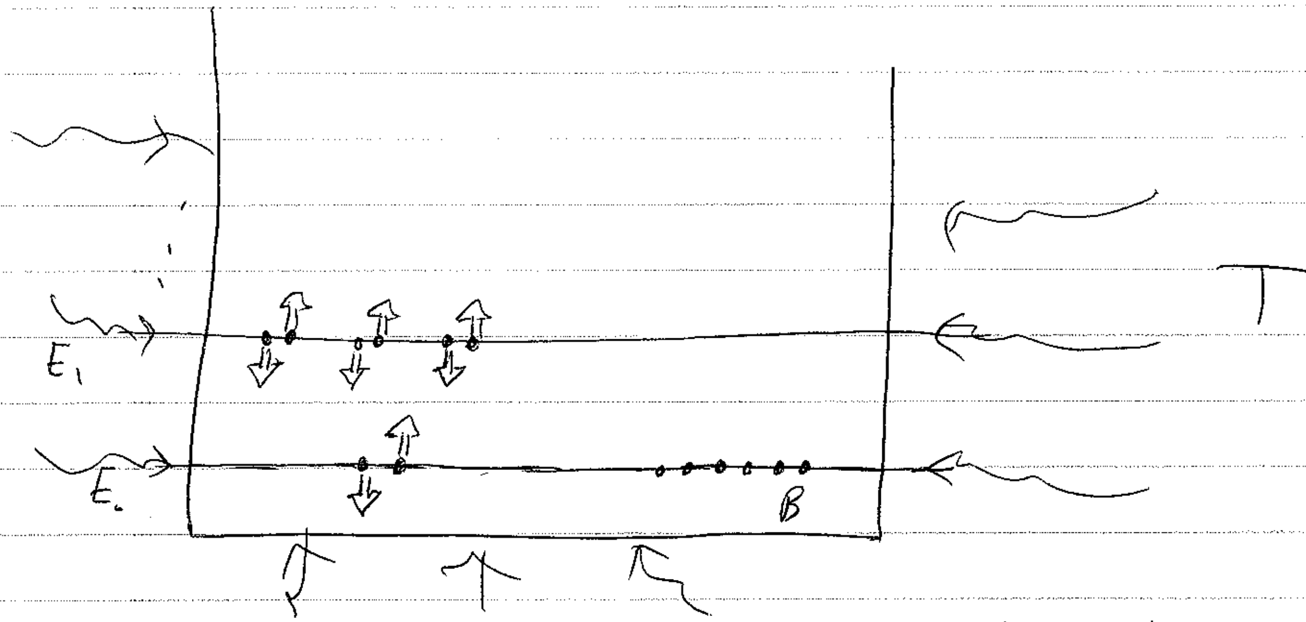
For a number N of particles in a volume V, the average distance follows;

$$d_{ave} \sim \left(\frac{V}{N}\right)^{1/3}$$

It therefore decreases as V decreases, and/or if N increases. This tells us that for small systems and/or dense systems, the quantum effects for identical particles become important. In fact, these effects have very interesting and important consequences.

Some Consequences of Identical Particles:

Consider one particle in a three-dimensional box of volume V. The ground state is unique with energy E_0, while the first (with energy E_1) excited level has triple degeneracy (for a cubic box);



At zero temperature the particle sits at the ground state.

At finite temperature, the particle can also occupy higher states and the distribution function obeys the Maxwell-Boltzman distribution.

Now lets add one more particle. At $T=0$, both particles are at the ground state, regardless of whether they are fermions or bosons. For fermions the spin degree of freedom allows the particles to be in the same orbital state while the total wave function is antisymmetric.

The situation will be different for three or more particles. For bosons, all the particles are at the ground state at $T=0$. However, the maximum number of fermions at the ground state is 2. So, other fermions occupy the first excited level (up to 6 fermions, due to triple degeneracy and spin degrees of freedom)

This implies that even at $T=0$, fermions occupy higher energy states. Depending on the number of fermions, energy levels up to an energy E_f are filled. E_f is called the "Fermi surface", since it represents a two-dimensional surface in the momentum space.

This is quite distinct from the classical case.

A fermi gas can have pressure even when $T=0$.

This has very important consequences, for example in "White dwarfs", where the zero-point pressure sustains the star against its gravitational attractive force.

For bosons, however, we can add as many particles as we want, and they all occupy the ground state at $T=0$. They form a condensate called "Bose-Einstein" condensate.

To see at what temperature the quantum effects become important (or unimportant), we can compare the average distance between particle d_{ave} with the thermal de Broglie wave length $\lambda_{\text{de Broglie}}^{\text{thermal}}$. The latter is given by:

$$\lambda \sim \frac{h}{\sqrt{3kmT}}$$

If the number of particles N is fixed, we see that $\lambda \sim d_{ave}$ (at some point) when temperature decreases. Below this temperature, quantum effects are important.

We can estimate $T_{critical}$ as:

$$\frac{h}{\sqrt{3kmT_{crit}}} \sim \left(\frac{V}{N}\right)^{\frac{1}{3}} \Rightarrow \boxed{T_{crit} \sim \frac{h^2}{3km}\left(\frac{N}{V}\right)^{\frac{2}{3}}}$$

On the other hand, at fixed temperature, $\lambda \sim d_{ave}$ at some point as we keep increasing the number of particles. This happens at $N_{critical}$, which is given by:

$$\left(\frac{V}{N_{crit}}\right)^{\frac{1}{3}} \sim \frac{h}{\sqrt{3kmT}} \Rightarrow \boxed{N_{crit} \sim \frac{V}{h^3}(3kmT)^{3/2}}$$

Therefore, quantum effects are important for:

(a) Small size, and/or, (b) Large number, and/or, (c) Low temperatures.

In such cases Maxwell Boltzmann distribution is no longer valid and one must use Bose-Einstein (in case of bosons) or Fermi-Dirac (in case of fermions) distributions.

The quantum effects amount to additional effective forces that are attractive in the case of bosons (so they can all be in the same state) or repulsive in the case of fermions (thus Pauli exclusion).

There are many physical consequences because of these distributions such as "Laser" (Bose-Einstein distribution) or "White dwarfs" (Fermi-Dirac distribution). The mere fact that we can have classical electromagnetic fields is a direct consequence of the photon being a boson.